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Lower Bounds on the Cluster Size Distribution

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We rigorously prove that the probability P_n that the origin of a d-dimensional lattice belongs to a cluster of exactly n sites satisfies $P_n > \exp(-\alpha n^{(d-1)/d})$ whenever percolation occurs. This holds for the usual (noninteracting) percolation models for any concentration $p > p_c$, as well as for the equilibrium states of lattice spin systems with quite general interactions. Such a lower bound applies also if no percolation occurs, but if it appears in some other phase of the system.

KEY WORDS: Percolation; Gibbs states; cluster size distribution; nucleation; stochastic geometry.

1. INTRODUCTION

1.1. Foreword

Percolation phenomena have recently been receiving an increasing amount of attention among physicists and mathematicians. Percolation has been used to describe a diverse collection of phenomena, including the percolation of a liquid through a porous medium, the sol–gel transition for polymers, and the magnetization of quenched ferromagnets. Definitions of various percolation models, their applications, and a review of recent results are given in Ref. 1.

Most often, independent percolation models are considered. More recently, interacting percolation problems have also been studied for the theory of nucleation[(2)] and for the theory of polymers.[(3)] It has also been recognized that the understanding of the stochastic geometry of the clusters may give other insights into the properties of phase transitions.[(4–6)]

The object of our study is the size distribution of finite clusters, which exhibits different asymptotic behavior in various regions of the parameter

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space. The lower bound which we prove depends mainly on the occurrence of percolation. As it applies equally well to systems with rather general interactions, we shall provide the proof in such a context.

1.2. Description of the Problem

The systems we consider are described by spin variables σ_x which are located at the sites x of the cubic lattice $\mathbb{Z}^d$, $d \geqslant 2$, and take the values ± 1. Alternatively, in lattice gas language one refers to the occupied or empty sites, which correspond to the two values of the spin.

The spins form a collection of random variables which for the usual (noninteracting) percolation model are independently distributed. However, one may also consider distributions which correspond to the thermodynamic equilibrium of an interacting spin system. An example of an interaction is given by the Hamiltonian function

$$H_\Lambda(\sigma) = -h \sum_{x \in \Lambda} \sigma_x - \sum_{\{x, y\} \cap \Lambda \neq \varnothing} J(x-y)\sigma_x \sigma_y \tag{1.1}$$

for spins in a region $\Lambda \subset \mathbb{Z}^d$. For finite systems, the probability of the configuration σ is given by $P_\Lambda(\sigma) = Z_\Lambda^{-1} \exp[-\beta H_\Lambda(\sigma)]$, where β is the reciprocal temperature and Z_Λ the appropriate normalization factor. If

$$\sum_{y \in \mathbb{Z}^d} |J(y)| < \infty$$

then it is well known that the system has a well-behaved thermodynamic equilibrium distribution. Boundary conditions determine possibly different phases in the thermodynamic limit. The system is said to be ferromagnetic if $J(y) \geqslant 0$ for all y.

The case of the noninteracting (usual) percolation model is recovered from this one by setting $J(y) = 0$ for all y and identifying positive spins with occupied sites, whose concentration is then $p = e^{+\beta h}/(e^{-\beta h} + e^{+\beta h})$.

For a given configuration, one may look for sets of plus spins which are connected through the bonds of the lattice. Maximal such sets will be called *clusters*. Hence a cluster is a set of occupied points connected through the bonds of the lattice and completely isolated by empty sites.

Let us now consider the event "the origin belongs to an infinite cluster" and denote its probability by P_∞. When $P_\infty > 0$, we say that percolation occurs. It is well known that when the sites are occupied independently with probability p, then percolation occurs for $p > p_c$ for some $0 < p_c < 1$, and not below. Percolation also occurs in the interacting problems; in particular for any integrable potential and any given temperature it appears at sufficiently large magnetic field. For the Ising ferromagnet at $h = 0$, percolation occurs in the plus phase whenever spontaneous magnetization exists. In fact, for the two-dimensional case the phase transition coincides with the appearance of an infinite cluster.[4] Percolation also exists for Ising models for any positive magnetic field when $\beta > \beta_c$.[7]

Let us denote by P_n the probability that the origin of the lattice belongs to a cluster of exactly n sites. P_n is the *cluster size distribution function*. One of the interesting problems arising in percolation theory is to determine the behavior of the cluster distribution function P_n with n. It is easy to see that $P_n < \exp(-\chi n)$ for some $\chi > 0$ at small concentration p in the independent case, and in the interacting case at large negative magnetic field. As a matter of fact, P_n decays exponentially for any p, except p_c, in the percolation model on a Bethe tree, as is well known from a work by Fisher and Essam.[8] An interesting question was then whether this exponential decay is always true outside the percolation thresholds.

1.3. Our Results and Previous Work

In contrast to the exponential decay at low concentrations, Stauffer[9] proposed in 1976, on the basis of an analysis of numerical studies, that $P_n > \exp(-\alpha n^{(d-1)/d})$, i.e., not exponentially, for $p > p_c$ in the two-dimensional ($d = 2$) independent case. Flamang[10] found this behavior in three dimensions. For the interacting case, Binder[11] argued and obtained numerical evidence for such a behavior in the low-temperature Ising model. For a very recent systematic numerical study of P_n, we refer to Ref. 12.

From the mathematical point of view, it was proved by Kunz and Souillard[13] that the moments $\langle |c|^k \rangle = \sum_n n^k P_n$ of the cluster size distribution satisfy $\langle |c|^k \rangle \geqslant [k \cdot d/(d-1)]!$ for any $p > p_c$, and that this inequality holds also for any ferromagnetic system in the whole percolative region. If P_n behaves as $\exp(-\chi n^{\xi})$, then this implies that $\xi \leqslant (d-1)/d$, i.e., the stronger inequality $P_n > \exp(-\chi n^{(d-1)/d})$.

Such a stronger statement was in fact derived in Ref. 13 in the case of the usual (noninteracting) percolation at sufficiently high concentration, but not in the whole percolative region. Upper bounds were also obtained there for p large enough, that is

$$\exp(-\alpha n^{(d-1)/d}) < P_n < \exp(-\alpha' n^{(d-1)/d})$$

Finally, analogous upper and lower bounds were recently proved by Delyon[14] for the case of the ferromagnetic d-dimensional Ising model in the positively magnetized phase at low temperature and at any temperature in a sufficiently high magnetic field. Such a behavior of the clusters of plus spins was also proved there in the minus phase at zero magnetic field and low temperature, which is a trace of nucleation at phase transition points.

The main result in this paper implies for the systems described above with a potential satisfying

$$\sum_{y \in \mathbb{Z}^d} |y||J(y)| < \infty$$

(and so in particular for independent percolation) that in the whole percolative region

$$P_n > \exp(-\chi n^{(d-1)/d})$$

Furthermore, such a bound also holds if there is no percolation but if percolation appears in some other phase, i.e., some other equilibrium state constructed with different boundary conditions.

Let us mention that one can prove a similar upper bound $P_n < \exp(-\chi' n^{(d-1)/d})$ in the large magnetic field region by extending easily the proof for the upper bound of Refs. 13 and 14.

Our results extend easily to other graphs as sketched in Ref. 13, and in particular to systems on the triangular and Kagomé lattices. In fact one only needs the existence of shapes whose volume grows faster than their surface, and the exponent of n represents the typical volume to surface scaling power.

2. THE RESULTS AND THE IDEA OF THE PROOF

In the previous section, we have introduced the collection of lattice spin variables. We denote by

$$\Omega = \{-1, +1\}^{\mathbb{Z}^d}$$

the space of all configurations σ of spins on the lattice $\mathbb{Z}^d$. For subsets $A \subset \mathbb{Z}^d$, Ω_A will denote the set of all spin configurations σ_A in A.

We shall consider probability measures μ on Ω, which give the distribution of the spin configurations σ, and refer to μ as the state of the system. Probabilities of sets $B \subset \Omega$ and expectation values of functions f will be denoted by $\mu(B)$ and $\mu(f)$. The states we shall consider correspond to thermodynamic equilibrium states for interactions which in finite volumes $\Lambda \subset \mathbb{Z}^d$ are described by

$$H_\Lambda(\sigma) = \sum_{\substack{A \subset \mathbb{Z}^d \\ A \cap \Lambda \neq \varnothing}} J_A \prod_{x \in A} \sigma_x \equiv h \sum_{x \in \Lambda} \sigma_x + \sum_{\substack{A \subset \mathbb{Z}^d \\ |A| \geqslant 2 \\ A \cap \Lambda \neq 0}} J_A \prod_{x \in A} \sigma_x \tag{2.1}$$

with some translation-invariant couplings J_A which include the one-body term, i.e., the magnetic field, $h \equiv J_{\{x\}}$.

For such interactions we introduce the norms

$$\|J\| = \sum_{A \ni 0} |J_A| \quad \left[\equiv \beta h + \beta \sum_y |J(y)| \text{ in physical notation} \right]$$

$$\|J\|_D = \sum_{A \ni 0} |J_A| \operatorname{Diam} A \quad \left[\equiv \beta h + \beta \sum_y |y| |J(y)| \right]$$

where Diam A is the diameter of the set A.

The probability distributions μ we consider are Gibbs states associated with a given interaction and are defined by the Dobrushin–Lanford–Ruelle equilibrium equations,[15] which state that, in any finite region $\Lambda \subset \mathbb{Z}^d$, the conditional probability distribution of σ_Λ given on external configuration η_{Λ^c} is

$$\exp[-H_\Lambda(\sigma_\Lambda \eta_{\Lambda^c})]/\text{norm} \tag{2.2}$$

where the temperature has been included in J.

The noninteracting distributions which are used for the standard site percolation problems are recovered by setting $J_A = 0$ whenever $|A| > 1$.

Our main results are:

Theorem 1. Let μ be a translation-invariant equilibrium (Gibbs) state for an interaction J with $\|J\|_D < \infty$. Then

$$P_\infty(\mu) > 0 \Rightarrow P_n(\mu) > a \exp(-cn^{(d-1)/d}) \tag{2.3}$$

with $c = 8 \cdot 3^{2d}(1 + 9d\|J\|_D)/P_\infty{}^2$ and some $a = a(P_\infty, d) > 0$.

Theorem 2. Let μ be a translation-invariant equilibrium (Gibbs) state for an interaction J with $\|J\|_D < \infty$. Then

$$P_\infty(\mu) > 0 \Rightarrow P_n(\mu') > a \exp(-c'n^{(d-1)/d}) \tag{2.4}$$

with $c' = 8 \cdot 3^{2d}(2 + 9d\|J\|_D)/P_\infty{}^2$ and some $a = a(P_\infty, d) > 0$, for any Gibbs state μ', not necessarily translation invariant, that corresponds to the same interaction.

Remarks. 1. The first theorem applies in particular to the usual percolation model for any $p > p_c$, and to the translation-invariant states of the Ising model whenever there is percolation. The second result yields, for example, the lower bound on P_n for low-temperature ferromagnets in the minus phase, where, however, plus spins do not percolate. It also yields the lower bounds for the non-translation-invariant states of the low-temperature, three-dimensional Ising model.

2. The explicit values for c are mentioned because of their generality; however, they do not yield the right behavior when P_∞ goes to zero. In

fact our lower bounds are obtained using clusters whose external boundary is rather regular. They give the correct qualitative behavior of P_n with n. However, when P_∞ goes to zero, clusters with an "ill-shaped" boundary become proportionally dominant, a phenomenon which we do not control here.

3. Another result of interest, which follows along the ideas of Ref. 13, using Lemma 4, yields constraints on the cluster size distribution: it says that for any Gibbs state μ corresponding to a potential J, $\|J\|_D < \infty$, one has

$$\frac{P_{\sum m_i}}{\sum m_i} \geqslant \sum_{\mathscr{C}_i, |\mathscr{C}_i| = m_i} \prod_i P(\mathscr{C}_i) \exp\left(-\alpha \sum_i |\partial^e \mathscr{C}_i| \right) \tag{2.5}$$

where the summation runs over all possible *shapes* of clusters of m_i sites, and $|\partial^e \mathscr{C}_i|$ denotes the length of the external boundary. In the case of independent percolation[13] or the ferromagnetic Ising model[14] one can prove a stronger result with $\alpha = 0$, i.e.,

$$\frac{P_{n+m}}{n+m} \geqslant \frac{P_n}{n} \frac{P_m}{m} \tag{2.6}$$

A weaker form of (2.5), Lemma 5, is used in the proof of Theorem 1. The constant c can perhaps be improved by a more complete partial summation over regular volumes in (2.5).

4. In the case when the interaction satisfies only $\|J\| < \infty$ and not $\|J\|_D < \infty$, we can still prove the analogs of Theorems 1 and 2, but where the lower bounds hold only for a subsequence of values of n which do not grow faster than geometrically.

5. Occasionally, the interactions (2.1) are given in terms of the "lattice gas" variables $n_A = \prod_{x \in A} (1 + \sigma_x)/2$. The same results hold if the analogous $\|\cdot\|_D$ is finite.

The Idea of the Proof

The reason for the particular power $(d-1)/d$ which appears in (2.3) may be seen by the following argument, which also yields the basic idea of the proof. When percolation occurs, then each point has the positive probability P_∞ of being connected to infinity by a path of plus spins. Therefore in any specified cube, the average fraction of the spins that are connected to the boundary is at least P_∞. It follows that the probability that a fraction of the spins larger than $P_\infty/2$ is connected to the boundary does not decrease to zero when the cube's size goes to infinity (in fact, it stays larger than $P_\infty/2$). For such configurations in the cube, it takes only a fluctuation of spins on the boundary, e.g., a formation of an outer layer of minus spins and an inner layer of plus spins, to form a cluster whose volume is neither more than the cube's volume nor less than the fraction $P_\infty/2$ of it. Since the above behavior of the spins inside the cube occurs spontaneously, the probability of such an event decays exponentially only with the cube's surface, while it provides a lower bound for the probability of having a cluster whose size is of the order of the volume. Thus $(d-1)/d$ appears as the minimal power for the given lattice by which the volume of a region has to be scaled to obtain the region's surface.

The proof of Theorem 1 is given in the following two steps:

(a) (Section 3). We show that for a sequence of values of n with an approximate geometrical progression, the lower bound (2.3) is already satisfied if one counts only clusters of cubic external boundary. This is done using the above idea, which, however, does not give the exact values of n for this sequence.

(b) (Section 4). Inequality (2.3) is proved for any n by using clusters with cubic external boundary, of sizes in the above-mentioned sequence, to build clusters of arbitrary size. It is important here to have an efficient method of construction, in which finite fractions of the volume and the external surface belong to the largest cluster.

Theorem 2 is a consequence of the previous analysis together with the fact that if several phases (Gibbs states) exist for a given interaction, that is, a given magnetic field, temperature, and potential, then it takes only surface energy to change the phase in some volume.

The general idea described above evolved from the basic work of Ref. 13, where it was discovered that when percolation occurs, clusters get an effective volume. However, the method used there did not yield the existence of a good sequence of n's in the whole percolative region even for the noninteracting case. The idea of using an almost geometric sequence to construct large clusters in an "efficient" way also appeared there and was used to get lower bounds on P_n for all n from those of a subsequence through the surmultiplicativity property (2.6). This yielded an alternative proof of step b for the case of independent spins and for the Ising model.[13,14]

3. LOWER BOUND FOR A SUBSEQUENCE

Let us introduce the standard cubes $\Lambda_k = [0, k-1]^d$. Given a region Λ, we denote by $\partial\Lambda$, $\delta\Lambda$, and $\theta(\Lambda)$ its outer boundary, inner boundary, and interior, which for the cubes are

$$\partial\Lambda_k = [-1, k]^d \backslash \Lambda_k, \qquad \theta(\Lambda_k) = [1, k-2]^d, \qquad \delta\Lambda_k = \Lambda_k \backslash \theta(\Lambda_k) \quad (3.1)$$

Consider the following events and the sets of configurations which represent them:

$$G_k^{\lambda} = \{\sigma \in \Omega | \text{the number of plus spins in } \theta(\Lambda_k) \text{ connected to } \delta\Lambda_k \text{ is larger than or equal to } \lambda|\theta(\Lambda_k)|\}$$

$$F_k^+ = \{\sigma \in \Omega | \sigma_x = +1 \text{ for all } x \in \delta\Lambda_k\}$$

We have:

Lemma 1. Let μ be a translation-invariant equilibrium state for an interaction J, with $\|J\| < \infty$. Then for any k

$$\mu(G_k^{P_\infty/2}) \geqslant P_\infty/2 \tag{3.2}$$

Proof. Let

$$f_\Lambda = \frac{1}{|\theta(\Lambda)|} \sum_{x \in \theta(\Lambda)} \chi_x^{\infty}$$

where χ_x^{∞} is the characteristic function of the event "the site x is connected to infinity," and for a finite set A, $|A|$ denotes the number of its points. For each configuration, f_Λ counts the fraction of the points in $\theta(\Lambda)$ that belong to an infinite cluster of plus spins. Now

$$P_\infty \equiv \mu(f_\Lambda) \leqslant P_\infty/2 + \int_{\{\sigma \in \Omega | f_\Lambda(\sigma) \geqslant P_\infty/2\}} f_\Lambda(\sigma)\, \mu(d\sigma) \tag{3.3}$$

$$\leqslant P_\infty/2 + \mu(\{\sigma \in \Omega | f_\Lambda(\sigma) \geqslant P_\infty/2\}) \tag{3.4}$$

since $\mu(f)$ denotes the average value of f and where use has been made of $f_\Lambda \leqslant 1$ in the second inequality. Therefore

$$\mu(G_k^{P_\infty/2}) \geqslant \mu(\{\sigma \in \Omega | f_\Lambda(\sigma) \geqslant P_\infty/2\}) \geqslant P_\infty/2 \tag{3.5}$$

where the first inequality expresses that there is a larger or equal number of points connected to the boundary $\delta\Lambda_k$ than to infinity. ■

Lemma 2. Let μ be a translation-invariant equilibrium state for an interaction J with $\|J\| < \infty$. Then for any k

$$\mu(G_k^{P_\infty/2} \cap F_k^+) \geqslant \tfrac{1}{2} P_\infty \exp(-K_1 |\delta\Lambda_k|) \tag{3.6}$$

with $K_1 = \ln 2 + 2\|J\|$.

Proof. Since

$$\mu(G_k^{P_\infty/2} \cap F_k^+) = \mu(G_k^{P_\infty/2}) \mu(F_k^+ | G_k^{P_\infty/2}) \tag{3.7}$$

where $\mu(A|B)$ denotes the conditional probability of A given B, the claim of the lemma is a consequence of Lemma 1 and the bound

$$\mu(F_k^+ | G_k^{P_\infty/2}) \geqslant \exp[-(\ln 2 + 2\|J\|)|\delta\Lambda_k|] \tag{3.8}$$

Inequality (3.8) is physically intuitive and follows directly through usual estimations once one realizes that the conditional expectation of F is a certain average of the expectations of F given specified conditions in $\mathbb{Z}^d \backslash \delta\Lambda_k$. ■

Let us now introduce for integers k, n the event

$$R_{k,n} = F_k^+ \cap \{\sigma \in \Omega | \text{the set of plus sites in } \Lambda_k \text{ connected to } \delta\Lambda_k \text{ has exactly } n \text{ points}\}$$

Then we have the following result:

Lemma 3. Let μ be a translation-invariant equilibrium state for an interaction J with $\|J\| < \infty$. *Then there exist sequences* $k(i)$, $n(i)$, $i = 1, 2,...,$ *such that*

(a) $$2^{d-1} \leqslant n(i+1)/n(i) < 3^d/P_\infty, \qquad n(1) = 1 \tag{3.9}$$

(b) $$n(i) \geqslant k(i)^d P_\infty/2 \tag{3.10}$$

(c) $$\mu(R_{k(i),n(i)}) \geqslant bn(i)^{-1} \exp[-K_2 n(i)^{(d-1)/d}] \tag{3.11}$$

with $b = (P_\infty/2)^2(1 - P_\infty/2)^{-1}$ and

$$K_2 = (\ln 2 + 2\|J\|)(2/P_\infty)^{(d-1)/d} \tag{3.12}$$

Proof. By definition, the set of configurations

$$G_k^{P_\infty/2} \cap F_k^+$$

is contained in the following disjoint union:

$$G_k^{P_\infty/2} \cap F_k^+ \subset \bigcup_{k^d P_\infty/2 \leqslant n \leqslant k^d} R_{k,n} \tag{3.13}$$

Hence

$$\mu(G_k^{P_\infty/2} \cap F_k^+) \leqslant \sum_{k^d P_\infty/2 \leqslant n \leqslant k^d} \mu(R_{k,n}) \tag{3.14}$$

which implies that for any given k there exists, in the interval $[P_\infty k^d/2, k^d]$, at least one integer $m(k)$ for which

$$\mu(R_{k,m(k)}) \geqslant (1 - P_\infty/2)^{-1} k^{-d} \mu(G_k^{P_\infty/2} \cap F_k^+) \tag{3.15}$$

Lemma 2 therefore ensures that

$$\mu(R_{k,m(k)}) \geqslant (P_\infty/2)(1 - P_\infty/2)^{-1} k^{-d} \exp(-K_1 k^{d-1}) \tag{3.16}$$

$$\geqslant (P_\infty/2)^2(1 - P_\infty/2)^{-1} m(k)^{-1} \times \exp[-K_1(2/P_\infty)^{(d-1)/d} m(k)^{(d-1)/d}] \tag{3.17}$$

using in the last step that $P_\infty k^d/2 \leqslant m(k)$. Statements (3.11) and (3.12) will follow from these estimates.

Finally one constructs the subsequence $n(i)$ satisfying (3.9) by induction starting from $n(1) = 1$: knowing $n(i)$, we choose $n(i+1)$ as the integer $m(k)$ in the interval $[P_\infty k^d/2, k^d]$ which satisfies (3.17), k being the smallest integer larger than or equal to $[2^d n(i)/P_\infty]^{1/d}$. ■

Remark 1. Consider the event

$$F_k^- = \{\sigma \in \Omega | \sigma_x = -1 \quad \text{for all} \quad x \in \partial\Lambda_k\}$$

Then the set $R_{k,n} \cap F_k^-$ corresponds to the event "there is a cluster of plus spins of exactly n sites whose boundary is the boundary of the cube." It is clear that

$$P_n \geqslant \mu(R_{k,n} \cap F_k^-) \tag{3.18}$$

However, simple estimates as in Lemma 2 ensure that

$$\mu(R_{k,n} \cap F_k^-) \equiv \mu(R_{k,n})\mu(F_k^- | R_{k,n}) \geqslant \mu(R_{k,n}) \exp(-K_1 |\partial\Lambda_k|) \tag{3.19}$$

Inequalities (3.18), (3.19) together with Lemma 3 imply then a lower bound of the expected form for a subsequence of values of n which has an approximate geometrical progression.

Remark 2. Up to this point we have only used that $P_\infty > 0$ and $\|J\| < \infty$. In the next section $\|J\|_D$ will be used.

4. PROOF OF THE LOWER BOUND FOR ALL *n*

We shall now use the subsequence constructed in Lemma 3 to build up clusters of arbitrary size, but before that we need a technical lemma which tells us how much a distribution in a cube depends on the outside.

Lemma 4. Let μ_1 and μ_2 be the Gibbs distributions in a box Λ for a given interaction J, with $\|J\|_D < \infty$, and given external spin configurations $\eta_{\Lambda^c}^{(1)}, \eta_{\Lambda^c}^{(2)}$. Then for any function $f \geqslant 0$ of the spin configuration in Λ

$$\mu_1(f) \geqslant \mu_2(f) \exp(-K_3 |\partial\Lambda|) \tag{4.1}$$

where $K_3 = 4\|J\|_D$.

Proof. First let us note the following bound on the interaction energy across the boundary of a box, Λ and Λ^c denoting, respectively, the box and its exterior:

$$\sum_{\substack{A \subset \mathbb{Z}^d : A \cap \Lambda \neq \varnothing \\ A \cap \Lambda^c \neq \varnothing}} |J_A| \leqslant \sum_l \sum_{\substack{x \in \Lambda \\ d(x, \partial\Lambda) \leqslant l}} \sum_{\substack{A \subset \mathbb{Z}^d \\ \mathrm{Diam}\, A = l \\ A \ni x}} |J_A| \leqslant \|J\|_D |\partial\Lambda| \tag{4.2}$$

The estimate of the lemma then follows with the use of (4.2) from the explicit relation

$$\mu_1(f) = \mu_2(f(\sigma_\Lambda) \exp[H_\Lambda(\sigma_\Lambda \eta^{(2)}_{\Lambda^c}) - H_\Lambda(\sigma_\Lambda \eta^{(1)}_{\Lambda^c}] \times \{\mu_2(\exp[H_\Lambda(\sigma_\Lambda \eta^{(2)}_{\Lambda^c}) - H_\Lambda(\sigma_\Lambda \eta^{(1)}_{\Lambda^c})]\}^{-1} \tag{4.3}$$

which is a consequence of (2.2). ■

Before going on to the proof of Theorems 1 and 2, we still need a technical lemma:

Lemma 5. Let $n(i)$ be the sequence constructed in Lemma 3 for a given interaction J. Then any integer n can be written as

$$n = \sum_{1 \leqslant i \leqslant i_{\max}} w_i n(i) \tag{4.4}$$

where $i_{\max}$ depends on n, and

$$0 \leqslant w_i \leqslant 3^d / P_\infty \tag{4.5}$$

Furthermore,

$$\sum_{1 \leqslant i \leqslant i_{\max}} w_i n(i)^{(d-1)/d} \leqslant K_4 n^{(d-1)/d} \tag{4.6}$$

with $K_4 = 4 \cdot 3^d / P_\infty$.

Proof. The property (4.4) is readily proved by repeatedly subtracting the maximal possible element of the sequence $n(i)$ from the remainder, starting with n. The bound (4.5) on the weights w_i follows from the property (3.9) of the sequence $n(i)$. Inequality (4.6) holds, since

$$\begin{aligned} &\sum_{1 \leqslant i \leqslant i_{\max}} w_i n(i)^{(d-1)/d} \\ &\leqslant \left(\sup_i w_i \right) \sum_{1 \leqslant i \leqslant i_{\max}} n(i_{\max})^{(d-1)/d} \left\{ \left(\sup_i \frac{n(i)}{n(i+1)} \right)^{(d-1)/d} \right\}^{i_{\max} - i} \\ &\leqslant \frac{3^d}{P_\infty} n^{(d-1)/d} \sum_{j \geqslant 0} 2^{-j(d-1)^2/d} \leqslant 4 \frac{3^d}{P_\infty} n^{(d-1)/d} \end{aligned} \tag{4.7}$$

and we have used that $d \geqslant 2$ for our value of K_4. ■

We shall now use the decomposition (4.4), $n = \sum_i w_i n(i)$, to select for each n a collection of w_i cubes of sides $k(i)$, where $k(i)$ is the sequence associated, by Lemma 3, to $n(i)$. There obviously is a systematic way of arranging the cubes so that (a) they are all disjoint, (b) the origin is a corner of one of them (say one of the largest), and (c) each cube is connected to the previous and consecutive ones by opposite corners (see Fig. 1). Let E_n be the event: "in each of the cubes thus placed, $\sigma_x = +1$ for all x on the inner boundary, and the number of plus spins inside each such cube of side $k(i)$

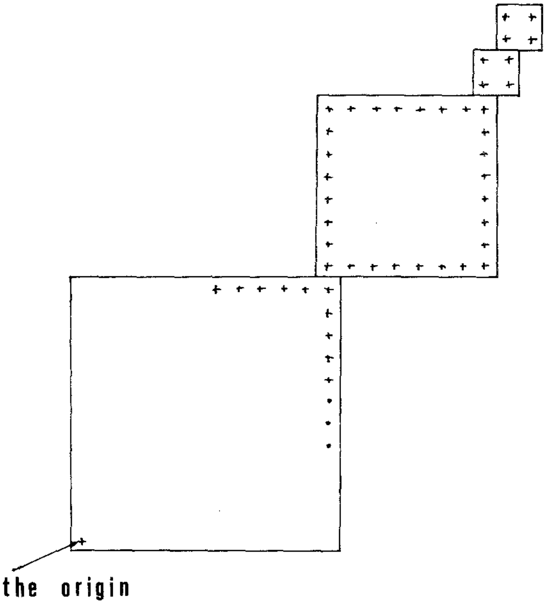

Fig. 1

that are connected to the boundary is exactly $n(i)$." Then we have:

Lemma 6. Let μ be a translation-invariant state for an interaction J with $\|J\|_D < \infty$. Then

$$\mu(E_n) > a \exp(-K_5 n^{(d-1)/d}) \tag{4.8}$$

for some $a = a(P_\infty, d)$ and $K_5 = 8(1 + 2\|J\| + 8d\|J\|_D)(3^d/P_\infty{}^2)$.

Proof. It follows from Lemmas 3 and 4 that

$$\mu(E_n) \geqslant \prod_{1 \leqslant i \leqslant i_{\max}} \{bn(i)^{-1} \exp[-K_2 n(i)^{(d-1)/d} - 2dK_3 k(i)^{d-1}]\}^{w_i} \tag{4.9}$$

and using (3.10) and (4.6), we get

$$\mu(E_n) \geqslant a \exp\{-K_4[K_2 + \epsilon + 2dK_3(2/P_\infty)^{(d-1)/d}] n^{(d-1)/d}\} \tag{4.10}$$

with $\epsilon = 1 - \ln 2$ and

$$a = \inf_{\{w_i\},j} \left\{ \prod_{1 \leqslant i \leqslant j} [bn(i)^{-1} e^{\epsilon n(i)}]^{w_i} \middle| |w_i| \leqslant 3^d/P_\infty,\ n(i)\text{—satisfying (3.9)} \right\}$$
$$= a(P_\infty, d) > 0 \quad \blacksquare \tag{4.11}$$

Proof of Theorem 1

Let us denote by V_n the volume constructed above by piling up cubes of appropriate size and ∂V_n the set of spins on its external boundary. Let us also denote by B_n^- the event "$\sigma_x = -1$ for $x \in \partial V_n$."

The event $E_n \cap B_n^-$ realizes clusters containing the origin and of exactly n plus spins. Hence it is clear that

$$P_n(\mu) \geqslant \mu(E_n \cap B_n^-) \tag{4.12}$$

But

$$\mu(E_n \cap B_n^-) = \mu(E_n)\mu(B_n^- | E_n) \tag{4.13}$$

and by the same estimate as in (3.8)

$$\mu(B_n^- | E_n) > \exp[-(\ln 2 + 2\|J\|)|\partial V_n|] \tag{4.14}$$

On the other hand we have

$$|\partial V_n| \leqslant \prod_{1 \leqslant i \leqslant i_{\max}} w_i[k(i) + 2]^{d-1} \tag{4.15}$$

Theorem 1 follows now from Lemma 6 and from (4.12)–(4.15), using (3.10), $n(i) \geqslant k(i)^d P_\infty/2$, and (4.6). ■

We shall now proceed to prove Theorem 2. It relies on the fact that if several phases coexist for a given interaction, that is, given magnetic field, temperature, and potential, then it takes only surface energy to change the phase in some volume.

Proof of Theorem 2

Let μ and μ' be as in Theorem 2. Integrating (4.1) over the boundary conditions in the states μ and μ' shows that

$$\mu'(E_n \cap B_n^-) \geqslant \mu(E_n \cap B_n^-) \exp\{-K_4|\partial[V_n \cup \partial V_n]|\} \tag{4.16}$$

where $\partial[V_n \cup \partial V_n]$ denotes the external boundary of the set $V_n \cup \partial V_n$. Theorem 2 follows now using (4.12), which is also valid for μ', and our previous bounds on $\mu(E_n \cap B_n^-)$. ■

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